MTH 531 Homework 3

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Problem 1.

Consider the discrete two-point set $\{0,1\}$. Prove that the product $\prod_{n=1}^{\infty}\{0,1\}$ is not discrete.

Solution. Let $U := \{(0,0,0,\ldots)\}$. We claim that U is not an open subset of $\prod_{n=1}^{\infty}\{0,1\}$, implying that $\prod_{n=1}^{\infty}\{0,1\}$ is not discrete.

Suppose towards a contradiction that U is open, and let Σ be the sub-basis which defines the product topology, that is,

$$\Sigma = \left\{p_j^{-1}(U_j) : j \in \mathbb{N}, U_j \subseteq^{op} \{0,1\}\right\}.$$

Since U is an open singleton, it must be an element of the basis $\mathcal{B}(\Sigma)$: in order to write U as the union of open sets $\{V_\alpha\}_{\alpha\in\Lambda} \subseteq \mathcal{B}(\Sigma)$, we must have that $V_\alpha \subseteq U$ for all $\alpha \in \Lambda$, so $V_\alpha = \emptyset$ or $V_\alpha = U$ for all $\alpha \in \Lambda$. If $V_\alpha = \emptyset$ for all $\alpha \in \Lambda$, then $U = \bigcup_\Lambda V_\alpha = \emptyset$, a contradiction; so some $\beta \in \Lambda$ has $V_\beta = U$ and thus $U \in \mathcal{B}(\Sigma)$.

Since $U \in \mathcal{B}(\Sigma)$, by the definition of a sub-basis, it can be written as the finite intersection

$$U = \bigcap_{k=1}^{n} p_{j_k}^{-1}(U_{j_k}) \quad \text{for some } j_1, j_2, \ldots, j_n \in \mathbb{N}.$$

Note, however, that by definition of the projection map we have, for all $i \in \mathbb{N}$,

$$p_i^{-1}(U_i) = \prod_{j=1}^{i-1}\{0,1\} \times U_i \times \prod_{j=i+1}^{\infty}\{0,1\}.$$

So we can take $M = \max\{j_k : 1 \le k \le n\}$ and see that

$$(0,0,\ldots,\underbrace{0},1,1,1,\ldots) \in \bigcap_{k=1}^{n} p_{j_k}^{-1}(U_{j_k}) = U,$$

where the underbrace indicates the Nth coordinate. But this contradicts the definition of U as a singleton containing only the sequence of 0s. So U is not open, and hence $\prod_{n=1}^{\infty}\{0,1\}$ is not discrete. □

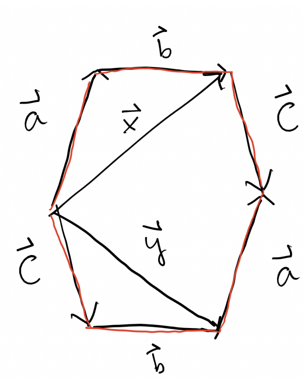
Problem 3.

Let X be the subset of the Euclidean plane consisting of the three polygons in the figure (two triangles and a quadrilateral). Let $\Sigma = X/\sim$ be the quotient space obtained from X by identifying boundary edges according to the labeling and orientation of those edges.

Part 1.

First identify the edges labeled x and y to convince yourself that Σ is homeomorphic to the surface defined by the word $abca^{-1}b^{-1}c^{-1}$ on the boundary of a hexagon.

Solution. Identifying x and y yields the following shape:

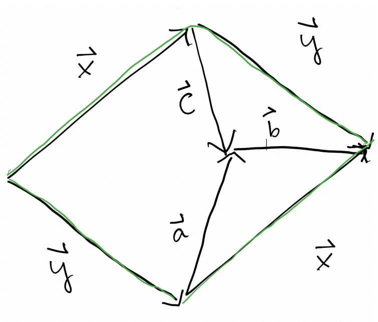


This is clearly a hexagon with identifications according to the word $abca^{-1}b^{-1}c^{-1}$. □

Part 2.

Starting with X again, perform the identifications in a different order to argue that Σ is homeomorphic to the surface defined by the word $xyx^{-1}y^{-1}$ on the boundary of a quadrilateral.

Solution. Identifying a, b, and c yields the following shape:



This is clearly a quadrilateral with identifications according to the word $xyx^{-1}y^{-1}$. □

Part 3.

Conclude that the words $abca^{-1}b^{-1}c^{-1}$ and $xyx^{-1}y^{-1}$ define homeomorphic surfaces.

Solution. Since the hexagon with identifications according to the word $abca^{-1}b^{-1}c^{-1}$ *and* the quadrilateral with identifications according to the word $xyx^{-1}y^{-1}$ are both homeomorphic to Σ, that homeomorphism defines an equivalence relation on topological spaces implies that the two spaces are homeomorphic. $\square$

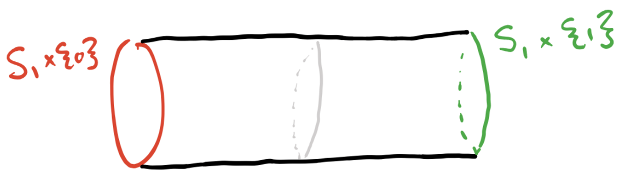
Problem 4.

Let P be the quotient space obtained from the two-sphere S^2 by identifying the north and south poles to a single point, as shown in the figure. Let Q be the quotient space obtained from the torus $S^1 \times S^1$ by identifying an entire slice $S^1 \times \{\text{pt}\}$ to a point, as shown in the figure. Show that P and Q are homeomorphic.

Solution. Let X be the cylinder $S_1 \times [0,1]$ subject to the following identification: we identify both $S_1 \times \{0\}$ and $S_1 \times \{1\}$ to a single point via a quotient map q, so that

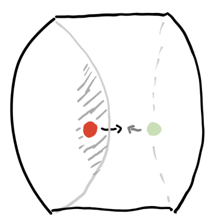
$$q(S_1 \times \{0,1\}) = \text{P}.$$

Visually, this looks like:

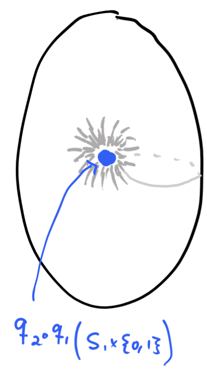


We write $q = q_1 \circ q_2 = q_3 \circ q_4$ by performing the identification in steps. First, we apply a quotient map q_1 to (a) identify $S_1 \times \{0\}$ to a point $Q_{\{0\}}$, and (b) identify $S_1 \times \{1\}$ to a point $Q_{\{1\}}$, yielding a surface Y_1 homeomorphic to a sphere. We then apply a second quotient map q_2 to identify $Q_{\{0\}}$ to $Q_{\{1\}}$:

(a) $Y_1 = q_1(X) \approx S_2$.



(b) The surface between Y_1 and $q_2(Y_1)$.



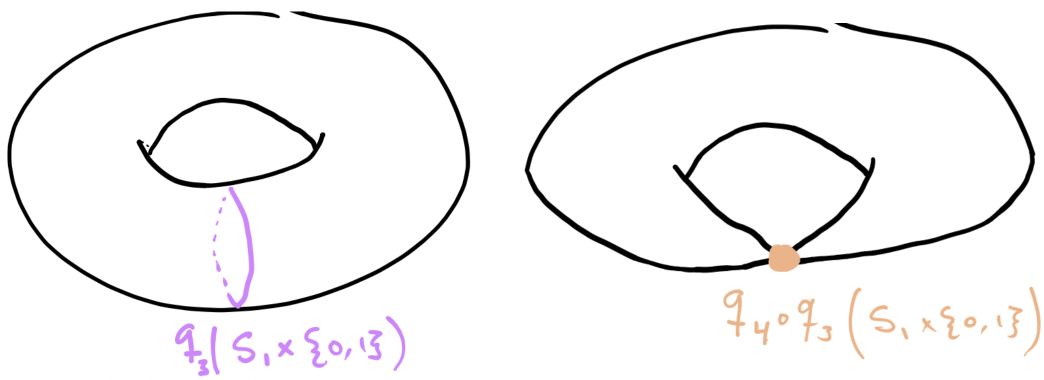
(c) $q(X) = q_2(Y_1)$.

This yields a "pincushion" shape where the sphere has had two antipodal points (its north and south poles) identified.

Now we perform the identifications in a different order: first we identify pairs of points on the edge of the cylinder via a quotient map q_3 satisfying

$$q_3(\{a\} \times \{0,1\}) = Q_{\{a\}} \quad \text{for all } a \in S_2.$$

This yields a surface Y_2 homeomorphic to the torus, $S_1 \times S_1$. We then apply a quotient map q_4 which identifies $Q_{\{a\}}$ for all a:



The resulting shape is the "pinched torus."

As in problem 3, $X = q(S_1 \times [0,1])$ is homeomorphic to *both* the pincushion *and* the pinched torus, and hence the two are homeomorphic since homeomorphism is an equivalence relation. □